Achromatic N-prism beam expanders: optimal configurations

Rick Trebino

In this paper, we calculate optimal prism configurations for achromatic N-prism beam expanders of a single material; we argue that for moderate to high magnifications, that is, $M \gtrsim [2 - 1/(2^{N-1} - 1)]^N$, the up-up . . . up-down configuration is generally optimal, in the sense that it maximizes the transmission for given magnification. We also derive exact expressions for the incidence and apex angles that optimize a nonachromatic N-prism beam expander of arbitrary materials. The use of simple three-prism (up-up-down) and four-prism (up-up-up-down) single-material achromatic beam expanders is suggested for applications requiring compactness, achromaticity, and temperature stability.

I. Introduction

Well known for over a century,[1] the prism beam expander has only recently found application as a laser-related optical device. In the past few years, researchers have employed the prism beam expander (PBE) to obtain 1-D beam expansion in situations requiring compactness, alignment simplicity, low cost, and/or a minimum of aberrations. In particular, the PBE has found application in wavemeters[2] and optical-fiber-diameter measurement schemes[3,4] and as a general-purpose laboratory tool. The most important application of the PBE, however, remains that of beam pre-expansion prior to diffraction by a grating.[5-22] Inside a high-gain laser cavity, this pre-expansion significantly improves laser linewidth at a small cost in laser efficiency. More importantly, the 1-D nature of the expansion maintains alignment simplicity—a tremendous advantage over spherical lenses, cylindrical lenses, or mirror telescopes, in which element spacings are critical or in which unnecessary 2-D expansion occurs. As a result, pulsed dye lasers[5-22] and occasionally CO_2 lasers[23] contain PBEs.

In many applications it is good practice to employ an achromatic PBE. While the dispersion of a nonachromatic PBE can be made to add to that of the grating and hence to further improve spectral resolution, the temperature-induced wavelength drift associated with the use of a nonachromatic PBE/grating combination in a laser cavity is often undesirable.[24] In single-axial-mode pulsed dye lasers, such temperature-dependent effects are unacceptable,[11,14,24] and as a result, commercial devices necessarily employ either an achromatic PBE[25] or temperature stabilization.[26] (Fejer *et al.*[3,4] were also obliged to avoid temperature effects and hence required approximate achromaticity in the PBE in their optical-fiber-diameter measurement device.) Finally, achromaticity allows extremely accurate sine-bar wavelength tuning of a laser employing a Littrow grating.

The simplest achromatic PBE consists of two prisms in a compensating up-down configuration so that the dispersion of the second prism precisely cancels that of the first. This configuration has been studied numerically by Barr,[27] who found that the use of very different prism apex angels can achieve achromaticity in such a device. Two-prism PBEs achieving moderate to large magnifications can be quite lossy, however, and as a result, greater numbers of prisms are often employed so that each prism incidence angle can be reduced (compared with incidence angles required in a two-prism device with comparable magnification) and the overall PBE transmission improved.[11,14] A four-prism achromatic beam expander, involving a compensating pair of compensating pairs (CPCP), i.e., a down-up-up-down configuration (see Fig. 1), of prisms has become popular and is used in several commercial pulsed dye lasers.[25] This device commonly achieves a magnification of the order of 40 with a single-pass transmission of >50%. Intuitively, this configuration is quite appealing, but is it optimal? In other words, does this arrangement yield the maximum transmission for given magnification for a four-prism achromatic PBE? And what configurations optimize achromatic PBEs with other numbers of prisms? And, in addition, is it pos-

The author is with Stanford University, Edward L. Ginzton Laboratory, Stanford, California 94305.
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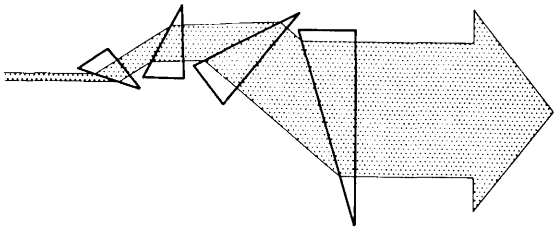

Fig. 1. Standard down–up–up–down (compensating pair of compensating pairs) achromatic four-prism beam expander. This device consists of two pairs of nearly achromatic two-prism beam expanders, with the second pair inverted with respect to the first. It achieves nearly collinear input and output beams but does not optimize transmission.

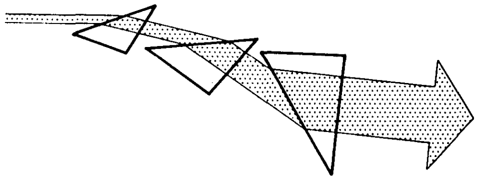

Fig. 2. Three-prism up–up–down beam expander. This configuration is optimal for achromatic three-prism beam expanders.

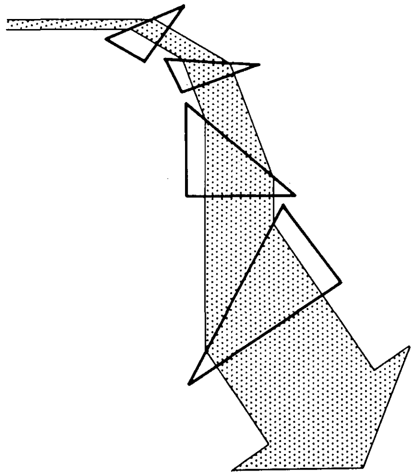

Fig. 3. Four-prism up–up–up–down beam expander. This configuration is optimal for achromatic four-prism beam expanders of a single material with total magnification $\gtrsim 10$ and probably also for lower magnifications.

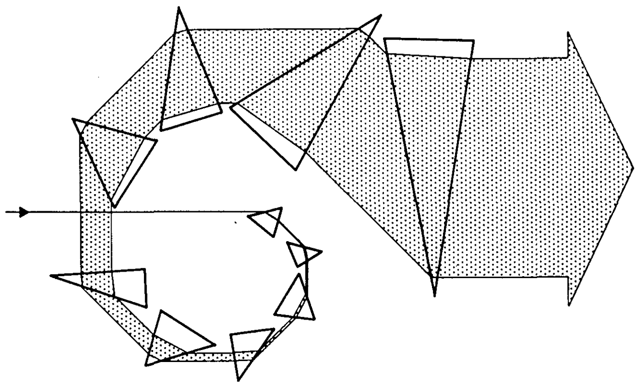

Fig. 4. N-prism up–up . . . up–down beam expander. This configuration is optimal for achromatic single-material prism beam expanders of moderate to large magnification, specifically, for magnifications greater than about $[2 - 1/(2^{N-1} - 1)]^N$. In addition, if each prism magnification is 2, such a device achieves a total magnification of 2^N, a dispersion of $1/2^{N-1}$ that of a single prism, and a transmission of 98% per prism.

sible to construct achromatic PBEs with odd numbers of prisms?

In this paper, we examine achromatic multiprism PBEs with regard to optimality, which we take to mean maximal transmission for given magnification. We show that, in general, any number of prisms (that is, except for 1) can produce an achromatic device, and interestingly, we find that the CPCP configuration is not in general the optimal achromatic four-prism beam expander. On the contrary, optimality is generally achieved by the somewhat unintuitive configurations in which the dispersions of the first $N - 1$ prisms add, with the dispersion of the last prism subtracting (i.e., an up–up . . . up–down configuration; see Figs. 2–4). In particular, we suggest a unique three-prism single-material achromatic beam expander, using an up–up–down configuration, which should be especially useful for pulsed dye lasers in which the cavity length must be kept short. For other applications, up–up . . . up–down configurations of higher numbers of prisms are suggested. We show that, for magnifications greater than about $[2 - 1/(2^{N-1} - 1)]^N$ (i.e., about 2^N for large N), these arrangements always optimize the performance of achromatic N-prism devices of a single material.

II. Optimal Nonachromatic *N*-Prism Beam Expanders

We begin by calculating exactly the optimal solution for the nonachromatic N-prism beam expander of arbitrary materials, the specific value, $N = 2$, having been solved approximately by Rácz *et al.*[15] for the case of identical material for both prisms. The results of this calculation will prove necessary for later achromatic PBE calculations.

Consider an N-prism beam expander composed of different materials with refractive indices, $n_1, n_2, \ldots, n_N$. We wish to determine the incidence angles and apex angles that maximize transmission for a given value of the total magnification, M, without concern for dispersion. Note, that, for now, the configuration (i.e., up–down, etc.) is irrelevant. We assume AR-coated exit faces, but since broadband AR coatings do not generally exist for very high incidence angles ($\gtrsim 65°$), we assume uncoated entrance faces. The transmission of each prism will be determined by reflective losses at the high-incidence-angle entrance face.

Let the ith prism have incidence angle θ_i and apex angle α_i, with additional angular quantities defined in Fig. 5. We take the refractive index of this prism to be $n_i = n_i(\lambda, T)$, where λ is the wavelength of the light incident on the prism and T is the temperature, which we take to be constant throughout the prisms. Finally, $\gamma_i = |\theta_i - \alpha_i + \nu_i|$ is the total deviation angle of an incident beam due to this prism (γ_i is here defined to be ≥ 0).

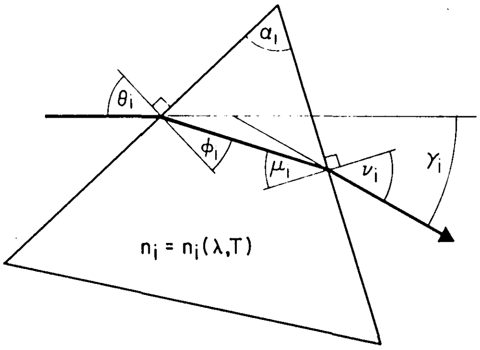

Fig. 5. Prism geometry.

The beam magnification due to the ith prism is[22]

$$M_i = \frac{\cos\phi_i}{\cos\theta_i}\cdot\frac{\cos\nu_i}{\cos\mu_i}, \tag{1}$$

and the total PBE magnification M is

$$M = \prod_{i=1}^{N} M_i. \tag{2}$$

Assuming a p-polarized beam, the transmission T_i of the ith prism will be[15]

$$T_i = \frac{4n_i \cos\phi_i/\cos\theta_i}{(n_i + \cos\phi_i/\cos\theta_i)^2}, \tag{3}$$

with the overall PBE transmission T given by

$$T = \prod_{i=1}^{N} T_i. \tag{4}$$

The reader is cautioned not to confuse the transmission with the temperature, both of which are denoted by the letter T.

Previous authors[7,13,15,17,18,24,27] have approximated the exit angle μ_i by the value zero and consequently M_i by the simpler result, $\cos\phi_i/\cos\theta_i$. We point out here, however, that this simplification can be deduced exactly for optimal designs. Clearly nothing is to be gained by allowing μ_i to be nonzero: nonzero values of μ_i act merely to decrease the magnification without changing the transmission. For nonachromatic PBEs, then, $\mu_i = 0$ is exactly optimal, and it is easy to show that the optimal values of all the system apex angles are

$$\alpha_i^{\text{opt}} = \phi_i^{\text{opt}}, \qquad i = 1,\ldots,N, \tag{5}$$

where ϕ_i^{opt} will be determined by θ_i^{opt}, which in turn will be determined by the optimal magnification.

We now observe that, with $\mu_i = 0$, the optimal individual prism magnification is exactly given by

$$M_i = \frac{\cos\phi_i}{\cos\theta_j}. \tag{6}$$

The individual prism transmissions can now be written in terms of M_i without approximation:

$$T_i = \frac{4n_iM_i}{(n_i + M_i)^2}, \tag{7}$$

so that the total transmission is

$$T = \frac{4n_1M_1}{(n_1+M_1)^2}\frac{4n_2M_2}{(n_2+M_2)^2}\cdots\frac{4n_NM_N}{(n_N+M_N)^2}. \tag{8}$$

Observe that the numerator of Eq. (8) is constant and equal to $4^N n_1n_2\ldots n_NM$, so that, for the purposes of optimization, we can ignore it. The problem thus reduces to minimizing the denominator, or equivalently, its square root. Thus we wish to minimize

$$L(M_i) = (n_1 + M_1)(n_2 + M_2)\ldots(n_N + M_N), \tag{9}$$

subject to the constraint

$$M_1M_2\ldots M_N = M. \tag{10}$$

A simple Lagrange multiplier calculation yields the optimal magnifications M_i^{opt}:

$$M_i^{\text{opt}} = \frac{n_i}{(n_1n_2\ldots n_N)^{1/N}}M^{1/N}, \qquad i = 1,\ldots,N, \tag{11}$$

which, when $n_1 = n_2 = \ldots = n_N$, reduce to $M_1^{\text{opt}} = M_2^{\text{opt}} = \ldots = M_N^{\text{opt}} = M^{1/N}$. The optimal incidence angles θ_i^{opt} are found to be

$$\theta_i^{\text{opt}} = \arcsin\left[\sqrt{\frac{(M_i^{\text{opt}})^2 - 1}{(M_i^{\text{opt}})^2 - 1/n_i^2}}\right], \tag{12}$$

so that the optimal apex angles α_i^{opt} are [using Eq. (5) and Snell's law]

$$\alpha_i^{\text{opt}} = \arcsin\left[\sqrt{\frac{(M_i^{\text{opt}})^2 - 1}{n_i^2(M_i^{\text{opt}})^2 - 1}}\right]. \tag{13}$$

Thus, for example, a PBE made entirely with material of refractive index 1.52 and achieving a magnification of 3 per prism should employ apex angles of 39.5° and incidence angles of 75.1°. The optimal transmission is easily obtained by substitution into Eq. (8) and, for the case of a single-material PBE, is $M[4n/(n + M^{1/N})^2]^N$. For the above example, the transmission will be 89% per prism. The transmission depends somewhat sensitively on the incidence angles, but the apex angles do not enter sensitively at all, with the transmission independent of these angles, and the magnification varying by <10% despite as much as a 20° variation in one of the apex angles. Thus, a very nearly optimal device can be achieved with apex angles quite different from α_i^{opt}.

III. Dispersion in Prism Beam Expanders: Discussion

The dispersion of the ith prism is[9]

$$\frac{\partial\gamma_i}{\partial n_i} = \frac{\sin\alpha_i}{\cos\phi_i\,\cos\nu_i}. \tag{14}$$

Dispersions of the form $\partial\gamma_i/\partial\lambda$ or $\partial\gamma_i/\partial T$ are easily obtained from Eq. (14) via the chain rule. We also define the orientation of the ith prism to be ϵ_i, with $\epsilon_i = +1$ indicating an up orientation, and $\epsilon_i = -1$ indicating a down orientation. Here, up and down refer to whether the ith prism bends the beam clockwise or counterclockwise, respectively, rather than to the prism's actual orientation in space (see Figs. 1–4). We now define $\gamma^{(N)}$ to be the total angular deviation of the N-prism PBE:

$$\gamma^{(N)} = \sum_{i=1}^{N} \epsilon_i\gamma_i \tag{15}$$

The total dispersion will be $\partial\gamma^{(N)}/\partial x$, where $x = n,\lambda$ or T:

$$\frac{\partial\gamma^{(N)}}{\partial x} = \sum_{i=1}^{N} \epsilon_i \frac{\partial\gamma_i}{\partial x} . \tag{16}$$

At a given wavelength and temperature, a PBE is considered achromatic to first order if $\partial\gamma^{(N)}/\partial\lambda = 0$ and thermally stable to first order if $\partial\gamma^{(N)}/\partial T = 0$. In fact, second-order variation of these quantities can cause a PBE, designed to be achromatic to first order at one wavelength, to have a nonzero value of $\partial\gamma^{(N)}/\partial\lambda$ at other wavelengths. This is an important point because PBEs in pulsed dye lasers can expect to see radiation from 300 nm to 1 μm in wavelength and beyond. Indeed, Barr[27] has shown that two-prism PBEs designed to have $\partial\gamma^{(2)}/\partial\lambda = 0$ at a given wavelength have varying amounts of dispersion at other wavelengths. A truly achromatic PBE would have zero-valued higher-order derivatives also: $\partial^k\gamma^{(N)}/\partial\lambda^k = 0$. Similarly, a truly thermally stable PBE would have zero-valued higher-order derivatives: $\partial^k\gamma^{(N)}/\partial T^k = 0$. (Alternatively, a designer could require first-order achromaticity or first-order thermal stability at two or more distinct operating points, simultaneously, which would also have the effect of minimizing the magnitudes of the higher-order derivatives.) The complexity of the problem makes it extremely difficult if not impossible to design a perfectly flat, zero-valued $\gamma^{(N)}$ vs λ or T curve, so perfect achromaticity or thermal stability is unlikely. Most applications, however, have not required extreme achromaticity or thermal stability, and first-order calculations have generally been sufficient.[27] For example, consider a single-axial-mode (0.001-Å linewidth) laser employing a Littrow grating and first-order-achromatic PBE designed for the wavelength λ_0 and temperature T_0. Now suppose that the laser operates at a wavelength quite distant from λ_0. Assuming a worst-case scenario, will the laser's wavelength now exhibit significant thermal drift? The grating dispersion will be $\sim 3 \times 10^3$ Å/rad; the prism material (take BK-7, for example) will have $|dn/dT| \approx 3 \times 10^{-6}$; and Barr[27] shows that, for first-order-achromatic two-prism designs, the rate of change of dispersion with respect to n, that is, $\partial^2\gamma^{(2)}/\partial n^2$, varies from ~ 0.8 to ~ 2 for reasonable prism parameters. Taking a rather large change in the prism refractive index (due to the variation in wavelength) of 0.015, resulting in $|\partial\gamma^{(2)}/\partial n| \lesssim 0.03$ at the new wavelength, we find that $|\partial\gamma/\partial T| \lesssim 3 \times 10^{-4}$ Å/°C. Thus, a (quite large) 3°C temperature change is now required to cause a wavelength drift of one linewidth. We conclude that first-order achromaticity is adequate in this case. That at least first-order achromaticity is required here is clear: the dispersion of a single beam-expanding prism, $\partial\gamma^{(1)}/\partial n$, is of order unity, resulting in a value of $|\partial\lambda/\partial T|$ about 30 times larger than that above. We will thus restrict our attention to first-order achromaticity or thermal stability, although it is probably a good idea to calculate the appropriate second-order quantity for a desired design. Such higher-order calculations are beyond the scope of this paper, however. In any case, we will continue to use the technically incorrect term "achromatic" as a shorthand for the longer "achromatic to first order at a given wavelength and temperature," hoping that this usage does not cause undue confusion.

For a PBE constructed from a single material of refractive index n, our interest will lie only in $\partial\gamma^{(N)}/\partial n$ because $\partial n/\partial$T or $\partial n/\partial\lambda$ will factor out of all terms of Eq. (16), and it is these remaining terms that we must set equal to zero. For a single-material PBE, then, achromaticity will be solely a geometrical consideration, with material considerations unimportant, except for a single parameter, the refractive index. Achromaticity for such a device, then, will not depend on possibly error-prone measurements of the material dispersion and will not be hurt by variations in this quantity from prism to prism. In addition, for such a device, achromaticity (i.e., wavelength-independent operation) will be equivalent to high temperature stability. This is the case because $\partial\gamma^{(N)}/\partial\lambda$ and $\partial\gamma^{(N)}/\partial T$ are both proportional to $\partial\gamma^{(N)}/\partial n$; the material dispersion ($\partial n/\partial\lambda$) or thermal derivative (dn/dT) factors out. For this reason, single-material achromatic PBEs are generally preferred.[28]

Duarte and Piper[20] show that

$$\frac{\partial\gamma^{(N)}}{\partial x} = \sum_{i=1}^{N} \frac{\epsilon_i \dfrac{\partial\gamma_i}{\partial x}}{\prod\limits_{j=i+1}^{N} M_j} . \tag{17}$$

More specifically, for two-, three-, and four-prism beam expanders,

$$\frac{\partial\gamma^{(2)}}{\partial x} = \frac{1}{M_2}\epsilon_1\frac{\partial\gamma_1}{\partial x} + \epsilon_2\frac{\partial\gamma_2}{\partial x} , \tag{18}$$

$$\frac{\partial\gamma^{(3)}}{\partial x} = \frac{1}{M_2M_3}\epsilon_1\frac{\partial\gamma_1}{\partial x} + \frac{1}{M_3}\epsilon_2\frac{\partial\gamma_2}{\partial x} + \epsilon_3\frac{\partial\gamma_3}{\partial x} , \tag{19}$$

$$\frac{\partial\gamma^{(4)}}{\partial x} = \frac{1}{M_2M_3M_4}\epsilon_1\frac{\partial\gamma_1}{\partial x} + \frac{1}{M_3M_4}\epsilon_2\frac{\partial\gamma_2}{\partial x} + \frac{1}{M_4}\epsilon_3\frac{\partial\gamma_3}{\partial x} + \epsilon_4\frac{\partial\gamma_4}{\partial x} . \tag{20}$$

Thus, the contribution of each individual prism to the total PBE dispersion is reduced by the total magnification experienced by the beam after that prism. This is why two identical prisms with identical incidence angles cannot produce an achromatic [i.e., $\partial\gamma^{(2)}/\partial x = 0$] PBE: different apex angles,[27] incidence angles, or materials for each prism are required—a situation somewhat analogous to the design of an achromatic doublet lens.

Equations (17)–(20) also illustrate a somewhat philosophical point about the achromatic PBE: that its construction from adjacent pairs of inverted, compensating prisms is neither necessary nor desired to attain achromaticity. To see this, note that only the final prism's dispersion contribution remains undiminished by any magnification-dependent factor. It is thus possible—indeed, probable—that, in high-magnification devices, the single dispersion term due to the final prism may dominate in Eqs. (17)–(20). As a result, all other prisms will have to be arranged to compensate for the dispersion of this single prism; otherwise severe design compromises will have to be made—reducing performance. The next section illustrates this point.

IV. Example

Consider an N-prism beam expander constructed from a very large number of prisms of the same material. Suppose that we desire an achromatic expander, and finally, suppose that we desire a total magnification of $M = 2^N$. We proceed now to construct an exactly optimal device for the case $N \to \infty$.

We have shown that to maximize the transmission of a single-material PBE, all prism magnifications should be equal, so we begin by setting $M_i = 2$ for all i. This relation determines all the prism incidence angles, and in conjunction with the use of normal exit angles from the prisms, determines the prism apex angles. So far, we have merely constructed the maximal-transmission PBE of magnification 2^N, paying no attention to the configuration or the device dispersion. It is likely that the above parameters cannot produce an achromatic PBE. In this example, however, the achromaticity constraint will prove satisfiable by appropriate choice of the various ϵ_i, that is, by appropriate choice of configuration. And clearly, since our choices of incidence and apex angles optimize the transmission of a PBE without regard to dispersion, this set of angles in conjunction with the achromatizing configuration will optimize the achromatic PBE of N-prisms with a magnification of 2^N as $N \to \infty$.

We proceed to determine this optimizing configuration by first observing that all θ_i are equal, and all α_i are equal. In addition, the exit angles ν_i, which are functions of θ_i, α_i, and n, are also equal. Equation (14) thus implies that all prism dispersions will be equal. Factoring out the individual prism dispersions, we find that the total PBE dispersion for this example is

$$\frac{\partial \gamma^{(N)}}{\partial n} \propto \epsilon_N + \frac{\epsilon_{N-1}}{2} + \frac{\epsilon_{N-2}}{4} + \frac{\epsilon_{N-3}}{8} + \ldots + \frac{\epsilon_1}{2^{N-1}}. \quad (21)$$

Remembering that $\epsilon_i = \pm 1$ depending on the prism orientation, we see that Eq. (21) sums to values between -2 and $+2$, depending on the ϵ_i. In the limit, $N \to \infty$, the series equals 0 exactly when $\epsilon_1 = \epsilon_2 = \epsilon_3 = \ldots = \epsilon_{N-1} = -\epsilon_N$, i.e., for the up–up–up ... up–down (or, equivalently, the down–down–down ... down–up) configuration (see Fig. 4 for $N = 10$). Other arrangements will not achieve achromaticity and will, as a result, require compromises, such as different magnifications (and hence less transmission), or different apex angles, which will result in less magnification and/or transmission. The above arrangement thus optimizes the achromatic N-prism beam expander with magnification 2^N in the limit of very large N.

This example appears somewhat contrived, but in fact it is actually quite representative of the problems encountered in the construction of achromatic PBEs.[29] Suppose, for example, that the PBE is to contain only a small number of prisms. In this case, the dispersion of the final prism will exceed that of all the preceding prisms combined. Of all possible configurations, the up–up ... up–down configuration, using $\theta_i = \theta_i^{opt}$ and $\alpha_i = \alpha_i^{opt}$, will most closely approach achromaticity, and will thus require the fewest design compromises, hence maximizing the transmission. Suppose, instead, that a greater magnification is required. Again, the up–up ... up–down configuration will most closely approach achromaticity with identical magnifications and apex angles and hence will require the fewest design compromises.

On the other hand, the example also illustrates situations in which the above configuration will not be optimal. Low-magnification expanders, requiring individual magnifications somewhat less than 2, will require additional prisms oriented in the same direction as the final prism, as the total dispersion contribution of the first $N - 1$ prisms will now exceed that of a similar final prism. In addition, the use of different materials for the various prisms could also mandate a different optimal achromatic configuration. Most useful achromatic PBEs will, however, consist of prisms of a single material to eliminate thermal and chromatic effects simultaneously. In addition, useful devices will generally demand a magnification per prism of ~2 or greater. Thus the up–up ... up–down arrangement will usually prove optimal in practice. The next section treats the optimal-configuration question more quantitatively.

V. Optimal Configurations for Achromatic *N*-Prism Beam Expanders of a Single Material

Analytical optimization of an achromatic N-prism beam expander requires the maximization of the transmission subject to two constraints: that the total magnification M is equal to a given value and that the total dispersion is zero. At its simplest, this problem is quite difficult, and Barr,[27] who took on the problem of the two-prism achromatic beam expander, did so numerically. The many-prism problem is further complicated by the large number of configurations that are possible even for intermediate numbers of prisms; for example, there are thirty-one possible nonredundant six-prism devices to consider. In this work, our goal is to ascertain the configuration appropriate to a given magnification for an N-prism achromatic device. We approach this problem by asking when the optimal solutions for nonachromatic PBEs will yield achromatic PBEs simply by proper choice of configuration. We will find that corresponding to each possible configuration will be a characteristic magnification for which that configuration yields an achromatic PBE with transmission equaling that of the optimal nonachromatic PBE. These values thus clearly determine the optimal achromatic PBE of the characteristic magnification. As a result, we will have obtained exact solutions to the problem of optimizing a single-material N-prism achromatic beam expander, but only for a few values of the magnification. Because the functions involved are continuous in the regions of interest, however, a configuration that is optimal at a given magnification will be the optimal configuration in a neighborhood of magnifications about the given magnification. The few optimal solutions derived here will then approximately determine the optimal configuration as a function of magnification, since devices requiring a magnification near one of the characteristic magnifications will necessarily employ the same configuration to achieve optimality.

Our approach then, remembering the results of Sec. II, is to assume equal magnifications and dispersions for each prism in the expander, with each individual prism magnification defined to be m and with the total magnification given by $M = m^N$. Factoring out the individual prism dispersion from each term in Eq. (17), setting the total dispersion $\partial\gamma^{(N)}/\partial n$ equal to zero, and multiplying through by m^{N-1}, we have the condition for simultaneous optimality and achromaticity:

$$\sum_{i=1}^{N} \epsilon_i m^{i-1} = 0, \tag{22}$$

a simple polynomial in the variable m. We will refer to Eq. (22) as the characteristic polynomial of the $\epsilon_1,\epsilon_2,\ldots,\epsilon_N$ configuration; there is a one-to-one correspondence between PBE configurations and polynomials with unity-magnitude coefficients. The root of a characteristic polynomial represents the individual prism magnification for which the $\epsilon_1,\epsilon_2,\ldots,\epsilon_N$ configuration is optimal.

To eliminate redundancy in the analysis that follows, we now define ϵ_N to be -1 so that the final prism points downward. There are thus 2^{N-1} possible nonredundant configurations for an N-prism beam expander, and rejecting the all-down configuration, which cannot be made achromatic, $2^{N-1} - 1$ possible configurations remain. There are thus three possible three-prism and seven possible four-prism nonredundant achromatic prism beam expanders. Roots of the characteristic polynomials were obtained using the SOLVE routine of an HP-15C calculator. Table I lists these roots for all three-, four-, five-, and six-prism beam-expander configurations. Imaginary roots and real roots that are less than one have been neglected, as they have no physical value here.

Observe that the up–up . . . up–down configuration always yields the highest magnification, and that this magnification approaches 2^N for large N, reminiscent of the example in Sec. IV. Configurations of the form down–up–up . . . up–down yield the next highest magnification, since the first prism's dispersion is reduced by the greatest number of magnification factors [see Eqs. (17)–(29)], so that its orientation matters the least. Note that many configurations have no physically interesting roots and hence are generally not useful choices for achromatic PBEs. In some cases, two configurations yield the same root, meaning that either can be employed to achieve the appropriate magnification optimally. These degeneracies occur because these configurations actually consist of pairs of smaller configurations. Specifically, the two six-prism expanders obtaining $M = 17.94$ actually contain two achromatic three-prism (up–up–down) beam expanders (one is inverted). The same holds true for the $M = 4.24$ pair of six-prism beam expanders: nonachromatic three-prism beam expanders appear as the even and odd prisms in these devices.

Table I. Exact Optimal Achromatic PBE Solutions

Number of prisms (N)	Configuration	m	$M = m^N$	T
3	*uud*	1.62	4.42	1.00
4	*uuud*	1.84	11.44	0.96
	duud, *udud*, *uudd*	1.00	1.00	0.85
5	*uuuud*	1.93	26.61	0.92
	duuud	1.72	15.15	0.98
	uduud	1.51	7.93	1.00
	uudud	1.29	3.58	0.97
	uuudd	1.18	2.28	0.93
6	*uuuuud*	1.97	57.73	0.89
	duuuud	1.88	44.60	0.93
	uduuud	1.79	33.16	0.95
	uuduud, *dduuud*	1.62	17.94	0.99
	uuudud	1.41	7.78	0.99
	uuuudd, *duduud*	11.27	4.24	0.96
	several	1.00	1.00	0.78

Note. Roots of the PBE characteristic polynomials and the resultant magnifications of exactly optimal achromatic PBEs of a single material. Here, m is the individual prism magnification, while M is the total magnification of the PBE. For each magnification shown here, the corresponding configuration is proved to be optimal (see text). Total transmissions T are listed assuming $n = 1.5$. Note that the up–up . . . up–down configuration, whose root m approaches 2 as $N \to \infty$, always yields the largest magnification. Use of individual prism magnifications less than the material refractive index n is wasteful; such low-magnification solutions, which require incidence angles below Brewster's angle, are included here only for completeness. (In this table, u = up and d = down.)

For magnifications other than those derived here, approximate optimal solutions should be obtainable by employing the configuration suggested by Table I, i.e., that of the listed magnification nearest to the desired magnification. Thus, a six-prism beam expander with a magnification of 42 (which is close to 44.60) should employ a down–up–up–up–up–down configuration for greatest transmission. We do not as yet know the precise angles to use for such a device, but the solution listed here serves as a good starting point toward finding this solution or approximating it empirically. Table I thus represents an approximate correspondence between the desired magnification and the optimal configuration for that magnification. This correspondence is illustrated in Fig. 6, which plots the optimal configuration for an achromatic PBE vs the desired magnification. Figure 7 illustrates (estimated) optimal transmission vs magnification curves for achromatic PBEs of various configurations with the exact optimal transmission vs magnification for a nonachromatic PBE also shown for reference.

From this analysis we conclude that any achromatic PBE with a desired magnification per prism approximately equal to or greater than the root of the characteristic polynomial of the up–up . . . up–down configuration (always the largest root) should employ that

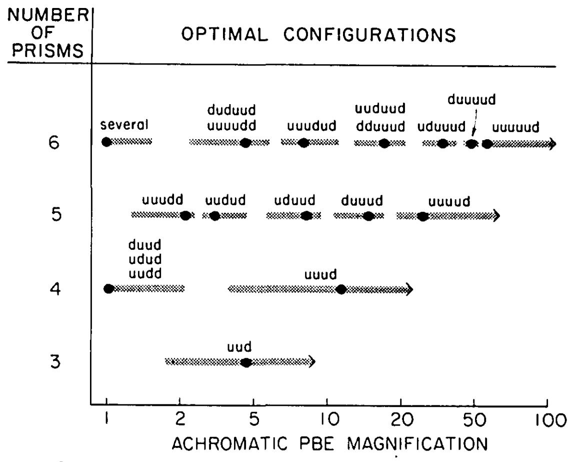

Fig. 6. Optimal prism configurations vs magnification for single-material three-, four-, five-, and six-prism achromatic beam expanders. Exact solutions derived herein are shown as dark dots, which are extended to indicate the estimated range of magnifications for which each configuration is optimal. The up–up . . . up–down configuration appears optimal for large magnifications for all PBEs.

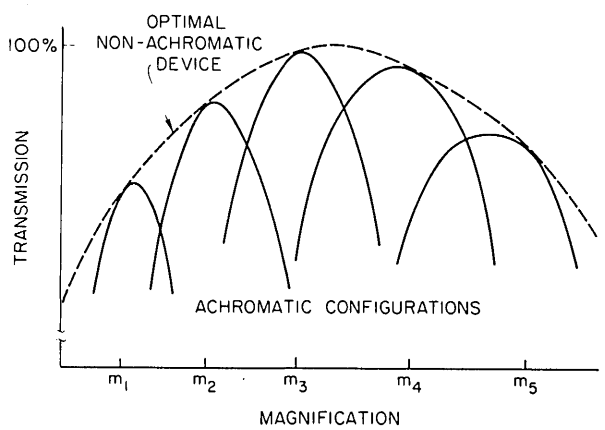

Fig. 7. Estimated optimal transmission vs magnification for several configurations for an N-prism single-material achromatic beam expander. Each configuration is optimal at some characteristic magnification and also in a neighborhood about that point but becomes nonoptimal near another configuration's characteristic magnification. (The horizontal axis can be interpreted as total magnification or magnification per prism). The dashed line illustrates the optimal transmission for a nonachromatic single-material PBE (see Sec. II), which attains maximal (100%) transmission for Brewster angle incidence, when the total magnification is n^N. Note that at the characteristic magnifications, $m_1, \ldots, m_5$, optimal nonachromatic transmission and optimal achromatic transmission (using the appropriate configuration) are equal.

configuration to obtain optimal performance. It is easy to show that this root is approximately given by $2 - 1/(2^{N-1} - 1)$. Since this root always corresponds to a magnification per prism of less than 2 and practical prism beam expanders employ larger magnifications, we conclude that in all practical situations the up–up . . . up–down configuration will be the optimal configuration. Precisely how much better this configuration will be than the down–up–up . . . up–down arrangement (the next best configuration) is not clear from our analysis, although we estimate that for magnifications near the up–up . . . up–down root, flipping the first prism will not decrease transmission significantly. This conclusion follows from the insensitivity of the magnification and transmission to small variations in the exit angle of the prism, which translates to insensitivity to apex angle. Thus, by varying prism apex angles, a nonoptimal configuration will probably achieve a transmission near to that of an optimal configuration for magnifications not too far from the nonoptimal configuration's characteristic magnification. For magnifications far from a configuration's characteristic magnification, the prism apex angles will necessarily deviate significantly from α_i^{opt}, prism exit angles will deviate significantly from zero, and demagnification at the exit face will become important. Higher incidence angles will be required to compensate for this demagnification in order to achieve the desired magnification, and as a result, the transmission will decrease significantly. Thus, for large magnifications, the up–up . . . up–down configuration will not only be optimal, but will be significantly more transmissive than other configurations. A quantitative resolution of this issue will appear in a future publication dealing with the special cases $N = 3$ and 4.

The up–up . . . up–down configuration may prove inconvenient for some applications due to the beam steering involved (see Figs. 2–4), but for other applications, such as preexpansion before a diffraction grating, this effect is generally not a problem since beam steering necessarily occurs anyway at the grating. The up–up–down three-prism and up–up–up–down four-prism achromatic PBEs may be of particular value for this application inside lasers, and in particular, in pulsed dye lasers. Replacement of current down–up–up–down devices with higher-transmission up–up–up–down devices (see Fig. 8) could improve dye laser efficiency. Alternatively, replacement with simpler up–up–down three-prism devices (see Fig. 9) will decrease dye laser cost and complexity. If the optimal achromatic up–up–down PBE proves less efficient than down–up–up–down devices of the same magnification, the increase in laser efficiency due to the shortening of the dye-laser cavity length by one prism may offset this loss. Finally, the decrease in the amount of glass in the cavity is desirable.

Another possibility is the use of the achromatic three-prism up–up–down device in a hybrid PBE/grazing-incidence dye-laser design, following the design of Rácz *et al.*,[15] who employed a two-prism nonachromatic beam expander in a grazing-incidence dye-laser cavity in order to decrease the high incidence angle of the grating to improve its efficiency from its value of ~5% at 89° incidence angle. Their design proved more efficient than either the simple grazing-incidence or four-prism Littrow-grating designs[25] at most useful linewidths. Since the diffraction efficiency of a 2400-lines/mm holographic grating in the visible remains relatively high (~70%) for incidence angles as high as ~70° and then trails off rapidly, a PBE magni-

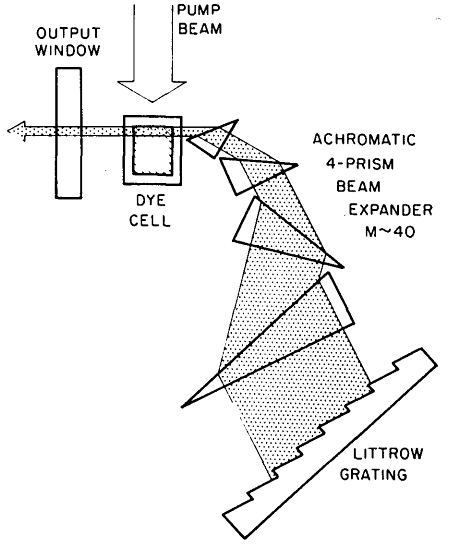

Fig. 8. Four-prism (up–up–up–down) achromatic PBE/Littrow-grating dye laser. The up–up–up–down configuration should be more efficient than the down–up–up–down configuration for relatively large magnifications, such as ~40.

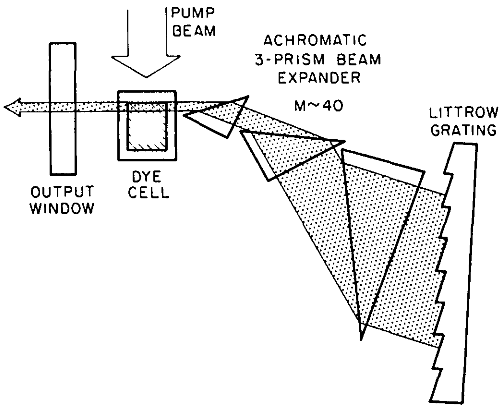

Fig. 9. Three-prism (up–up–down) achromatic PBE/Littrow-grating dye laser. For a magnification of ~40, a transmission of ~50% can be achieved.

fication of ~20 in such a hybrid design will continue to fill an ~70° incidence angle, 5-cm grating (maintaining good linewidth) and, assuming high PBE transmission, will yield very efficient dye-laser operation. A three-prism achromatic up–up–down glass PBE with a magnification of 20 will transmit >70% of the light incident on it[30] and hence should prove ideal for such a hybrid dye-laser design (see Fig. 10). At present we are operating such a three-prism hybrid dye laser pumped by a frequency-doubled Nd:YAG laser, and with it we are achieving much higher efficiencies than we have with a simple grazing-incidence arrangement. Due to its significantly higher efficiency, and the weak tolerances on all prism orientation angles, its alignment is easier than that of the grazing-incidence dye laser, and the beam bending due to the prisms does not prove problematic. Our application of this laser required only ~1-GHz wavelength stability, so we do not here report thermal-stability data.[31]

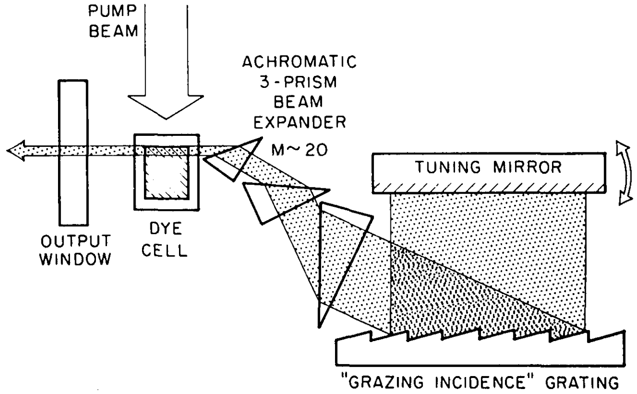

Fig. 10. Three-prism (up–up–down) achromatic PBE/grazing-incidence-grating dye laser. Such a hybrid design is more efficient than the PBE/Littrow-grating design or the grazing-incidence design.[15] Use of a three-prism achromatic design (vs the nonachromatic two-prism design of Ref. 15) should allow thermally stable single-mode operation and should further improve efficiency. (In Figs. 8–10, the pump beam is shown entering from above for diagrammatic simplicity only.)

VI. Conclusions

We have obtained an exact solution to the problem of maximizing the transmission of an N-prism nonachromatic beam expander of arbitrary materials and have employed this solution to obtain exact optimal solutions for single-material achromatic N-prism beam expanders of various magnifications. From these results we have shown that the value of the magnification determines the optimal configuration of an achromatic N-prism beam expander, and we have tabulated these configurations for achromatic three-, four-, five-, and six-prism single-material beam expanders. We argued that for practical achromatic PBEs, the up–up . . . up–down configuration always maximizes the transmission and hence is to be preferred in most applications. In particular, three-prism (up–up–down) and four-prism (up–up–up–down) single-material achromatic devices may be of great use, especially in pulsed dye lasers.

Our analysis of achromatic single-material PBEs, which was based on finding zeros of polynomials with unity-magnitude coefficients, can be easily generalized to treat the more general problem of multimaterial PBEs by simply allowing more general coefficients. Our analysis does not, however, indicate exact incidence and apex angles for general achromatic prism beam expanders. In addition, it gives no indication precisely how superior one configuration is over another. These two issues will be the subject of a future paper in which numerical solutions for three- and four-prism achromatic beam expanders will be reported.

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28. We note here that intracavity laser applications, for example, generally require high temperature stability but could actually benefit from some dispersion, which could act to further decrease the laser linewidth somewhat. Thus a device with $\partial\gamma^{(N)}/\partial T \approx 0$ at all relevant wavelengths but with large $\partial\gamma^{(N)}/\partial\lambda$, constructed necessarily of more than one material could be quite useful. Whether the appropriate materials exist to fabricate such a device is unknown to this author.
29. The arrangement of this example may actually be of practical interest. Using a finite number of identical glass ($n = 1.5$) prisms, each with a magnification of 2, yields a device with magnification 2^N, dispersion equal to $1/2^{N-1}$ of that of a single prism (perhaps small enough for many applications), and a reflection loss of only 2% per prism.
30. A far from optimal three-prism achromatic expander can be constructed easily and cheaply from already coated off-the-shelf 45–45–90 BK-7 prisms, yielding a magnification of 20.1 and a transmission of 65%. The required incidence angles are 80°, 77°, and 53°, respectively. The use of smaller apex angles will more closely approach optimality and hence will yield higher transmission. The above nonoptimal case is probably of practical value, however.
31. The question of thermal stability due to PBE dispersion is greatly complicated by the issue of the thermal stability of the mechanical mounts used for the optics in the cavity which can cause as much as $\sim$0.5 $cm^{-1}/°C$ drift in the dye-laser wavelength.[32] The use of thermally stable or compensated construction for such mounts is critical even in multimode devices. Taking such care, we have observed mechanical mount-induced thermal drifts of <0.1 $cm^{-1}/°C$ near room temperature. Commercial designs, in general, do even better.
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